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A Generalization of Ito's Formula

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Introduction

In [1], Ito's formula for the Brownian motion is extended to the functions whose Laplacian (in the sense of the distributions) is a measure (cf. [4] for a more probabilistic version). This case seems to be the most general one which can be achieved by the finite dimensional methods. A further generalization of these results would be the case in which the functions are replaced by the distributions; however, evaluation of a distribution at a point of the Euclidean space on which it is defined is meaningless, but there is a concept in the distribution theory which is analogous to the evaluation of the functions at a point, that is, the evaluation of the convolution of a distribution with the Dirac measure corresponding to that point at any $\mathscr{C}_0^\infty$-function and this is the main idea which has been to the origin of this work.

We start by treating the simplest case, i.e., $B=(B_t; t \geqslant 0)$ is a d-dimensional standard Brownian motion ($d \geqslant 1$) and the distributions belong to $\mathscr{S}'(\mathbb{R}^d)$, i.e., the space of the tempered distributions on $\mathbb{R}^d$. With the help of the Ito's formula, we calculate explicitly $T * \delta_{B_t}$, where $T \in \mathscr{S}'(\mathbb{R}^d)$, δ_{B_t} is the Dirac measure concentrated at B_t and $*$ represents the convolution, as a sum of a local martingale and a process of finite variation with values in $\mathscr{S}'(\mathbb{R}^d)$ (cf. [9, 10, 12]). The second section covers the proof of the existence and the uniqueness of the solutions of the heat equation in $\mathscr{S}'(\mathbb{R}^d)$, and Section III, the results of the first section are extended to the arbitrary $\mathbb{R}^d$-valued semimartingales and to the elements of $\mathscr{D}'(\mathbb{R}^d)$, i.e., the space of the distributions on $\mathbb{R}^d$.

I. A Generalization of Ito's Formula

We represent by $(\Omega, \mathscr{F}, P)$ a completed probability space and by $(\mathscr{F}_t; t \in \mathbb{R}_+)$ an increasing, right continuous filtration of $\mathscr{F}$ such that $\mathscr{F}_0$ contains all the P-negligible subsets of Ω. If $B=(B_t; t \in \mathbb{R}_+)$ is a standard

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d-dimensional Brownian motion, δ_{B_t} defines an $\mathscr{S}'(\mathbb{R}^d)$-valued semimartingale (cf. [9, 12]). Since as a distribution, $\delta_{B_t(\omega)}$ is of compact support, the convolution $T * \delta_{B_t(\omega)}$ is a well-defined distribution for any $(t, \omega) \in \mathbb{R}_+ \times \Omega$ and it can be expressed as

$$(T * \delta_{B_t})(\phi) = (T(\xi) \otimes \delta_{B_t}(\eta))(\phi(\xi + \eta)) = T(\phi(\cdot + B_t)),$$

where ξ and η denotes the space parameters in $\mathbb{R}^d$ with respect to which the distributions T and δ_{B_t} are defined (cf. [7]) and $\phi \in \mathscr{S}(\mathbb{R}^d)$. Now we announce:

THEOREM I.1. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$, then one has the following relation*:

$$T * \delta_{B_t} = T - \int_0^t \nabla T * \delta_{B_s} \cdot dB_s + \frac{1}{2} \int_0^t \Delta T * \delta_{B_s} \, ds$$

where $\nabla T * \delta_{B_t}$ denotes $(\partial T * \delta_B / \partial x_1, \ldots, \partial T * \delta_B / \partial x_d)$ and the stochastic integral is defined as

$$\left(\int_0^t \nabla T * \delta_{B_s} \cdot dB_s \right)(\phi) = - \sum_{i=1}^d \int_0^t T * \delta_{B_s} \left(\frac{\partial \phi}{\partial x_i} \right) dB_s^i$$

for $B_t = (B_t^1, \ldots, B_t^d)$.

Remark. Since the mapping $x \to \tau_x \phi$ defined as $\tau_x \phi(y) = \phi(x + y)$ for $\phi \in \mathscr{S}(\mathbb{R}^d)$ is continuous with values in $\mathscr{S}(\mathbb{R}^d)$, the process $T * \delta_{B_t}$ has weakly continuous trajectories and $\mathscr{S}'(\mathbb{R}^d)$ being a $\mathscr{DF}$-space, $T * \delta_{B_t}$ is locally bounded (cf. [9]), hence the stochastic integral is well defined.

We shall prove the theorem in several steps. For the sake of simplicity we suppose $d = 1$. First, if $\phi \in \mathscr{S}(\mathbb{R})$, by Ito's formula (cf. [3]) we have

$$\phi(\xi + B_t) = \phi(\xi) + \int_0^t \phi'(\xi + B_s) \, dB_s + \frac{1}{2} \int_0^t \Delta\phi(\xi + B_s) \, ds.$$

The mapping defined by $(t, \omega) \to \phi(\cdot + B_t(\omega))$ is a stochastic process with values in $\mathscr{S}(\mathbb{R})$, hence $\phi(\xi + B_t) - \phi(\xi)$ has also the same property. Moreover $\xi \to \frac{1}{2} \int_0^t \Delta\phi(\xi + B_s) \, ds$ belongs also to $\mathscr{S}(\mathbb{R})$ since

$$\left| \xi^k D_\xi^p \int_0^t \psi(\xi + B_s) \, ds \right| \leqslant |\xi|^k \int_0^t |D_\xi^p \psi(\xi + B_s)| \, ds \xrightarrow[|\xi| \to \infty]{} 0 \qquad \text{a.e.}$$

for any $p, k \in \mathbb{N}$. Consequently $(t, \omega) \to \int_0^t \phi'(\cdot + B_s) \, dB_s$ is a stochastic process with vaues in $\mathscr{S}(\mathbb{R})$, and we can write

$$(T * \delta_{B_t})(\phi) = T(\phi) + T\left(\int_0^t \phi'(\cdot + B_s) \, dB_s \right) + \frac{1}{2} T\left(\int_0^t \Delta\phi(\cdot + B_s) \, ds \right).$$

If we can show that $T \in \mathscr{S}'(\mathbb{R})$ commutes with the integrals, the theorem would be proved. From now on $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$ will be denoted respectively by $\mathscr{S}$ and $\mathscr{S}'$.

LEMMA I.1. *For any $T \in \mathscr{S}'$ and $\psi \in \mathscr{S}$, one has the following relation*:

$$T\left(\int_0^t \psi(\cdot + B_s)\, ds\right) = \int_0^t T(\psi(\cdot + B_s))\, ds \qquad \text{a.e. for any } t \geqslant 0.$$

Proof. Define T_n as $T_n = \inf\{t: |B_t| > n\}$, then T_n is an $\mathscr{F}_t$-stopping time and $T_n \uparrow +\infty$ as n tends to infinity. If $\psi \in \mathscr{S}$, then the process $(t, \omega) \to \psi(\cdot + B_{t\wedge T_n}(\omega)) \equiv \psi(\cdot + B_t^n(\omega))$ is bounded. Let K_n be the closed absolutely convex hull in $\mathscr{S}$ of the set

$$\{\psi(\cdot + B_t^n(\omega)) \in \mathscr{S}: (t, \omega) \in \mathbb{R}_+ \times \Omega\}.$$

Since $\mathscr{S}$ is a Montel space, K_n is compact in $\mathscr{S}$; denote by $\mathscr{S}[K_n]$ the completion of the subspace spanned by K_n with respect to the norm $g_{K_n}(\phi) = \inf\{\lambda > 0: \phi \in \lambda K_n\}$. Let K_n^0 be the polar of K_n in $\mathscr{S}'$, and denote by $\mathscr{S}'(K_n^0)$ the completion of the quotient space $\mathscr{S}'/r_{K_n^0}^{-1}(0)$ with respect to the seminorm $r_{K_n^0}$ which is the gauge function of K_n^0, i.e., if $F \in \mathscr{S}'$, then $r_{K_n^0}(F) = \inf\{\lambda > 0: F \in \lambda K_n^0\}$. It is well known that $(\mathscr{S}[K_n], \mathscr{S}'(K_n^0))$ forms a dual pair (cf. [6]) and if π_n denotes the canonical mapping from $\mathscr{S}'$ onto $\mathscr{S}'(K_n^0)$, then for any $T \in \mathscr{S}'$, $\phi \in \mathscr{S}[K_n]$, one has

$$\langle \phi, \pi_n(T) \rangle = T(\phi),$$

where $\langle \cdot, \cdot \rangle$ is the bilinear form associated to the dual pair $(\mathscr{S}[K_n], \mathscr{S}'[K_n^0])$.

The stochastic process $(t, \omega) \to \psi(\cdot + B_t^n(\omega))$ is weakly continuous, adapted with values in $\mathscr{S}[K_n^0]$, moreover it is bounded. Hence the Lebesgue integral converges in $\mathscr{S}[K_n^0]$. If $T \in \mathscr{S}'$, then

$$\begin{aligned}\left\langle \pi_n(T), \int_0^t \psi(\cdot + B_s^n)\, ds \right\rangle &= \int_0^t \langle \pi_n(T), \psi(\cdot + B_s^n) \rangle\, ds \\ &= \int_0^t T(\psi(\cdot + B_s^n))\, ds.\end{aligned}$$

Moreover. in [9], we have showed that this result is independent of the choice of the compact set K_n absorbing $\psi(\cdot + B^n)$; since $T_n \uparrow +\infty$, the lemma is proved. Q.E.D.

We have the analogous result for the stochastic integral:

LEMMA I.2. *For any $\psi \in \mathscr{S}$ and $T \in \mathscr{S}'$, one has the following relation*:

$$T\left(\int_0^t \psi(\xi + B_s)\, dB_s\right) = \int_0^t T(\psi(\cdot + B_s))\, dB_s \qquad \text{a.e. for any } t \geqslant 0.$$

Proof. With the notation of the proof of Lemma I.1, we have

$$\begin{aligned} T\left(\int_0^t \psi(\xi + B_s^n)\, dB_s\right) &= \left\langle \pi_n(T), \int_0^t \psi(\cdot + B_s^n)\, dB_s \right\rangle \\ &= \int_0^t T(\psi(\cdot + B_s^n))\, dB_s \end{aligned}$$

since $\psi(\cdot + B^n)$ is bounded, predictable with values in $\mathscr{S}[K_n]$. Moreover, as it is shown in [9, Theorem II.4], this result is independent of the choice of K_n. Q.E.D.

Now, the proof of Theorem I.1 is easy to see by writing

$$\begin{aligned} (T * \delta_{B_t})(\phi) &= T(\phi) + \int_0^t T\left(\frac{\partial \phi}{\partial x}(\cdot + B_s)\right) dB_s + \frac{1}{2}\int_0^t T\left(\frac{\partial^2 \phi}{\partial x^2}(\cdot + B_s)\right) ds \\ &= T(\phi) - \int_0^t \frac{\partial T * \delta_{B_s}}{\partial x}(\phi)\, dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 T * \delta_{B_s}}{\partial x^2}(\phi)\, ds. \end{aligned}$$

II. HEAT EQUATION

In this section using Theorem I.1 we study the heat equation in $\mathscr{S}'$. First let $T \in \mathscr{S}'$ such that

$$E\int_0^t |T * \delta_{B_s}(\phi)^2|\, ds < +\infty \qquad \text{for} \quad t \geqslant 0, \;\; \phi \in \mathscr{S}.$$

Then $\int_0^t T * \delta_{B_s}\, dB_s$ is a martingale with values in $\mathscr{S}'$. Taking the expectation of $T * \delta_{B_t}$ we find, for any $\phi \in \mathscr{S}$, that

$$\begin{aligned} H_t(\phi) \equiv E[T * \delta_{B_t}(\phi)] &= T(\phi) + \frac{1}{2}\int_0^t E[\Delta T * \delta_{B_s}(\phi)]\, ds \\ &= T(\phi) + \frac{1}{2}\int_0^t E[T * \delta_{B_s}(\Delta\phi)]\, ds \\ &= T(\phi) + \frac{1}{2}\int_0^t \Delta H_s(\phi)\, ds, \end{aligned}$$

hence H_t, defined as $E[T * \delta_{B_t}(\phi)] = H_t(\phi)$, is a solution of the heat equation

$$\frac{\partial U}{\partial t} = \frac{1}{2} \Delta U, \qquad U(0) = T \in \mathscr{S}'.$$

In fact, using an extension of Ito's formula we will prove a uniqueness result. To do so, we need the following result:

LEMMA II.1. *Suppose that $H: \mathbb{R} \to \mathscr{S}'(\mathbb{R}^d)$ is a continuous mapping of class $\mathscr{C}^1$, i.e., for any $\phi \in \mathscr{S}(\mathbb{R}^d)$, the function $t \to H_t(\phi)$ is of class $\mathscr{C}^1$. Then one has the following relation*:

$$H_t * \delta_{B_t} = H_0 - \int_0^t \nabla_x H_s * \delta_{B_s} \cdot dB_s + \int_0^t \left[\frac{dH_s}{ds} * \delta_{B_s} + \frac{1}{2} \Delta_x H_s * \delta_{B_s} \right] ds. \tag{II.1}$$

Proof. Let T be a positive number and denote by $\mathscr{X}$ the completion of $\mathscr{C}_0^\infty((0, T))$, i.e., the infinitely differentiable functions on $[0, T]$, whose supports are in $(0, T)$, with respect to the seminorm defined as

$$f \to \|f\|^2 = \int_0^T |f(t)|^2 \, dt + \int_0^T \left| \frac{df(t)}{dt} \right|^2 dt.$$

Then, the closed graph theorem implies that the mapping $\phi \to H(\phi)$ is continuous on $\mathscr{S}(\mathbb{R}^d)$ with values in $\mathscr{X}$, so it is a nuclear mapping. Hence it can be represented as (cf. [6])

$$H_t = \sum_{i=1}^\infty \lambda_i F^i \otimes f_i(t) \qquad dt\text{-a.e.},$$

where $(\lambda_i) \in l^1$, $\{f_i\}$ is a bounded sequence in $\mathscr{X}$ and $\{F^i\}$ is equicontinuous in $\mathscr{S}'(\mathbb{R}^d)$. By Theorem I.1, $H_t * \delta_{B_t}$ can be written in the differential form as

$$\begin{aligned} d(H_t * \delta_{B_t}) &= \sum_i \lambda_i [F^i * \delta_{B_t} \, df_i(t) + f_i(t) \, d(F^i * \delta_{B_t})] \\ &= \sum_i \lambda_i \left[F^i * \delta_{B_t} \frac{df_i}{dt} \, dt - f_i(t) \nabla_x F^i * \delta_{B_t} \cdot dB_t \right. \\ &\quad \left. + \frac{1}{2} f_i(t) \Delta_x F^i * \delta_{B_t} \, dt \right] \\ &= - \sum_i \lambda_i f_i(t) \nabla_x F^i * \delta_{B_t} \cdot dB_t + \sum_i \lambda_i \frac{df_i(t)}{dt} F^i * \delta_{B_t} \, dt \\ &\quad + \frac{1}{2} \sum_i \lambda_i f_i(t) \Delta_x F^i * \delta_{B_t} \, dt. \end{aligned}$$

Hence for an arbitrary representation of H we have

$$H_t * \delta_{B_t} = H_0 - \sum_i \lambda_i \int_0^t f_i(s)\, \nabla_x F^i * \delta_{B_s} \cdot dB_s + \sum_i \lambda_i \int_0^t \frac{df_i(s)}{ds} F^i * \delta_{B_s}\, ds$$
$$+ \frac{1}{2} \sum_i \lambda_i \int_0^t f_i(s)\, \Delta_x F^i * \delta_{B_s}\, ds.$$

The proof will be completed if we show that the sum commutes with the integrals. In order to see this, stop (B_t) as in the proof of Lemma I.1. Then it is not difficult to see that the integral

$$\int_0^{t \wedge T_n} \sum_i \lambda_i \nabla_x F^i * \delta_{B_s} f_i(s) \cdot dB_s$$

converges in $\mathscr{S}'(\mathbb{R}^d) \tilde{\otimes} L^2(\Omega, \mathscr{F}, P)$, i.e., the completed projective tensor product of $\mathscr{S}'(\mathbb{R}^d)$ and $L^2(\Omega, \mathscr{F}, P)$. For the other integrals, the proof is similar and for any $\phi \in \mathscr{S}(\mathbb{R}^d)$ we have

$$H_t * \delta_{B_t}(\phi) = H_0(\phi) - \int_0^t \nabla_x H_s * \delta_{B_s}(\phi) \cdot dB_s$$
$$+ \int_0^t \left[\frac{dH_s}{ds} * \delta_{B_s}(\phi) + \frac{1}{2} \Delta_x H_s * \delta_{B_s}(\phi) \right] ds$$

$dP \times dt$ – a.e., but the right-hand side is continuous in t hence this relation is true up to an evanescent process; since T is arbitrary, the proof is completed.

Q.E.D.

Remark. Instead of $\mathscr{K}$ one can choose the completion of $\mathscr{C}_0^\infty((0, T))$ with respect to the seminorm

$$f \to \sup_{0 \leqslant t \leqslant T} \left[|f(t)| + \left| \frac{df(t)}{dt} \right| \right]$$

and with this choice the representation of H_t, holds for all t and the proof becomes simpler.

In order to prove the uniqueness of the solution of the heat equation, we shall use the following result:

LEMMA II.2. *Let $T \in \mathscr{S}'(\mathbb{R}^d)$ be such that*

$$E\left[\int_0^t (T * \delta_{B_s}(\phi))^2\, ds \right] < +\infty \qquad \textit{for any} \quad t \in \mathbb{R}_+, \quad \phi \in \mathscr{S}(\mathbb{R}^d),$$

then for any $\phi \in \mathscr{S}(\mathbb{R}^d)$

$$E\left[\int_0^t (H_{a-s} * \delta_{B_s}(\phi))^2\, ds\right] < +\infty \qquad \textit{for any} \quad t \leqslant a, \;\; da - \textit{a.e.},$$

where H_a *is defined as* $H_a(\phi) = E[T * \delta_{B_a}(\phi)]$.

Proof. Without loss of generality, we may suppose that (B_t) is a Markov process with respect to the filtration $(\mathscr{F}_t)_{t \geqslant 0}$. Moreover, we can take T in the form $D_\xi^\beta g$ where g is a continuous polynomially bounded function (cf. [5, p. 139]) and D_ξ^β represents the differential operator $D_{\xi_1}^{\beta_1} \cdots D_{\xi_d}^{\beta_d}$ with $\beta_1 + \cdots + \beta_d = \beta$ and $D_{\xi_i}^{\beta_i} = \partial^{\beta_i}/\partial \xi_i^{\beta_i}$. Then, if $\phi \in \mathscr{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}&\int_0^t (H_{a-s} * \delta_{B_s}(\phi))^2\, ds \\ &\qquad = \int_0^t \left| \int_{\mathbb{R}^d} g(\xi)\, E_{\tilde{\omega}}[D_\xi^\beta \phi(\xi + B_{a-s}(\tilde{\omega}) + B_s(\omega))]\, d\xi \right|^2 ds, \qquad \text{(II.2)}\end{aligned}$$

where $E_{\tilde{\omega}}$ represents the mathematical expectation with respect to $\tilde{\omega}$. If we denote by E_x the mathematical expectation with respect to Wiener measure W with $W\{B_0 = x\} = 1$, (II.2) can be written

$$\begin{aligned}&\int_0^t \left| \int_{\mathbb{R}^d} g(\xi)\, E_{B_s(\omega)}[D_\xi^\beta \phi(\xi + B_{a-s})]\, d\xi \right|^2 ds \\ &\quad = \int_0^t \left| \int_{\mathbb{R}^d} g(\xi)\, E_0[D_\xi^\beta \phi(\xi + B_a) \mid \mathscr{F}_s]\, d\xi \right|^2 ds \\ &\quad \leqslant \int_0^t E_0\left[\left| \int_{\mathbb{R}^d} g(\xi)\, D_\xi^\beta \phi(\xi + B_a)\, d\xi \right|^2 \,\Big|\, \mathscr{F}_s \right] ds \\ &\quad = \int_0^t E_0[|T * \delta_{B_a}(\phi)|^2 \mid \mathscr{F}_s]\, ds.\end{aligned}$$

Consequently one has

$$E \int_0^t (H_{a-s} * \delta_{B_s}(\phi))^2\, ds \leqslant tE[|T * \delta_{B_a}(\phi)|^2] < +\infty \qquad da\text{-a.e. Q.E.D.}$$

Now the uniqueness can be proved:

THEOREM II.1. *Suppose that* $T \in \mathscr{S}'(\mathbb{R}^d)$ *satisfies the following relation*:

$$E \int_0^t |T * \delta_{B_s}(\phi)|^2\, ds < +\infty$$

for any $\phi \in \mathscr{S}(\mathbb{R}^d)$. *Then the mapping* $H: R_+ \to \mathscr{S}'(\mathbb{R}^d)$ *defined as* $H_t(\phi) = E[T * \delta_{B_t}(\phi)]$ *is a solution of*

$$\frac{dH}{dt} = \frac{1}{2} \Delta H, \qquad t > 0,$$

$$H_0 = T.$$

Moreover, if G *is another solution with weakly differentiable trajectories then,* $G = H$.

Proof. The existence has been proved at the beginning of this section. The uniqueness goes on as in the finite dimensional case: By Lemma II.1 for any $a > 0$

$$H_{a-t} * \delta_{B_t} = H_a + M_t^a,$$

where M_t^a is a local martingale; but by Lemma II.2, it is da-almost everywhere a martingale. Taking the mathematical expectation we find for $t = a$

$$E[T * \delta_{B_a}] = H_a, \qquad da\text{-a.e.}$$

hence $G_a = H_a$ da-a.e.; moreover they have the weakly continuous trajectories and the uniqueness follows. Q.E.D.

Remark. In fact this result can be extended to those G whose trajectories are almost everywhere differentiable.

III. Extensions to Semimartingales

In this section we shall work again in the one-dimensional cases; however the results can be extended trivially to any finite dimensional Euclidean space.

Let X be a semimartingale. If $\phi \in \mathscr{S}$, then one has (cf. [3])

$$\begin{aligned} \phi(X_t + \xi) = \phi(X_0 + \xi) &+ \int_0^t \phi'(X_{s-} + \xi)\, dX_s \\ &+ \frac{1}{2} \int_0^t \phi''(X_{s-} + \xi)\, d\langle X^c, X^c \rangle_s \\ &+ \sum_{0 < s < t} [\phi(\xi + X_s) - \phi(\xi + X_{s-}) - \phi'(\xi + X_{s-}) \Delta X_s], \end{aligned}$$

where X_{t-} represents the left limit of X_t at t and $\Delta X_s = X_s - X_{s-}$. We have

LEMMA III.1. *Let*

$$A_t(\omega, \xi) = \sum_{0 \leqslant s \leqslant t} [\phi(\xi + X_s) - \phi(\xi + X_{s-}) - \phi'(\xi + X_{s-}) \Delta X_s],$$

then $(t, \omega) \to A_t(\omega, \cdot)$ *is an adapted, right continuous stochastic process with values in* $\mathscr{S}$.

Proof. Let $K_t(\omega) = \sup_{s \leqslant t} |X_s(\omega)|$. Then $X_s(\omega) \in [-K_t(\omega), K_t(\omega)]$ and $\Delta X_s(\omega) \in [-2K_t(\omega), 2K_t(\omega)]$ for any $s \leqslant t$. Hence the set

$$M_t(\omega) = \{\phi(\cdot + \Delta X_s(\omega)) : 0 \leqslant s \leqslant t\}$$

is relatively compact in $\mathscr{S}$. If q is any continuous seminorm on $\mathscr{S}$, by the Taylor's formula, we have

$$q(A_t(\omega, \cdot)) \leqslant \sum_{s \leqslant t} (\Delta X_s)^2 \, q'(\phi(\cdot + \theta \Delta X_s(\omega))), \qquad \theta \in [-1, 1],$$

where q' is also a continuous seminorm. Hence

$$q(A_t(\omega, \cdot)) \leqslant \sup\{q'(\phi) : \phi \in M_t(\omega)\} \cdot \sum_{s \leqslant t} (\Delta X_s)^2 < +\infty;$$

consequently $A_t(\omega)$ converges in $\mathscr{S}$ for almost everywhere ω. Measurability follows from the fact that the mapping

$$(t, \omega, \xi) \to \phi(\xi + X_t(\omega))$$

is measurable since it can be written a

$$((t, \omega), \xi) \to (X_t(\omega), \xi) \circ y \to \phi(y). \qquad \text{Q.E.D.}$$

THEOREM III.1. *For any tempered distribution T one has the following relation*:

$$T * \delta_{X_t} = T * \delta_{X_0} - \int_0^t T' * \delta_{X_{s-}} \, dX_s + \frac{1}{2} \int_0^t T'' * \delta_{X_{s-}} \, d\langle X^c, X^c \rangle_s$$
$$+ \sum_{0 \leqslant s \leqslant t} [T * \delta_{X_s} - T * \delta_{X_{s-}} + T' * \delta_{X_{s-}} \Delta X_s],$$

where X^c *represents the continuous local martingale part of* X.

Proof. By Lemma III.1 the term corresponding to the jumps of X is well defined; for the other terms, one proceeds exactly as in the proofs of Theorem I.1 and Lemma I.1. Q.E.D.

Remark. Let us note that this result extends almost trivially to $\mathscr{D}'$, i.e., the space of the distributions on $\mathbb{R}$ (hence to $\mathscr{D}'(\mathbb{R}^d)$). To see this it is sufficient to remark that the mapping

$$x \to \phi(\cdot + x) = \tau_x \phi$$

is continuous from $\mathbb{R}$ into $\mathscr{D}$ for any $\phi \in \mathscr{D}$; hence the mapping $(t, \omega) \to \tau_{X_t(\omega)}(\phi)$ is measurable and (with the notations of Lemma III.1) $M_t(\omega) = \{\phi(\cdot + \Delta X_s(\omega)): 0 \leqslant s \leqslant t\}$ is a bounded set in $\mathscr{D}$.

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